

# A four-person chess-like game without Nash equilibria in pure stationary strategies

November 4, 2014

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## Abstract

In this short note we give the example of a four-person finite positional game with perfect information that has no positions of chance and no Nash equilibria in pure stationary strategies. The corresponding directed graph has only one directed cycle and only five terminal positions. It remains open:

- (i) if the number  $n$  of the players can be reduced from 4 to 3,
- (ii) if the number  $p$  of the terminals can be reduced from 5 to 4, and most important,
- (iii) whether it is possible to get a similar example in which the outcome  $c$  corresponding to all (possibly, more than one) directed cycles is worse than every terminal for each player.

Yet, it is known that

- (j)  $n$  cannot be reduced to 2,
- (jj)  $p$  cannot be reduced to 3, and
- (jjj) there can be no similar example in which each player makes a decision in a unique position.

**Keywords:** stochastic, positional, chess-like, transition-free games with perfect information and without moves of chance; Nash equilibrium, directed cycles (dicycles), terminal position.

## 1 Introduction

In this paper we answer in the negative to one of the two open questions suggested in [3]. This negative result was conjectured in [30]. We refer the reader to these two papers for definitions of the chess-like and backgammon-like games, and of Nash-solvability, which we recall here just briefly.

The *backgammon-like* and *chess-like* games are finite positional  $n$ -person games with perfect information, which can and, respectively, cannot have random moves.

More precisely, such a game is modeled by a finite directed graph (digraph)  $G = (V, E)$ , whose vertices are partitioned into  $n + 2$  subsets:  $V = V_1 \cup \dots \cup V_n \cup V_T \cup V_R$ . A vertex  $v \in V_i$  is interpreted as a *position* controlled by the player  $i \in I = \{1, \dots, n\}$ , while  $v \in V_R$  is a positions of chance, with a given probabilistic distribution on the outgoing edges; finally,  $v \in V_T = \{a_1, \dots, a_p\}$  is a *terminal position* (or just a *terminal*, for short), which has no outgoing edges. Furthermore, a directed edge  $(v, v')$  is interpreted as a *move* from the position  $v$  to  $v'$ . We also fix an initial position  $v_0 \in V \setminus V_T$ .

A game is called *chess-like* if it has no positions of chance,  $V_R = \emptyset$ .

The digraph  $G$  may have *directed cycles* (dicycles). Recall that positions may be repeated in a backgammon or in a chess play. We assume that all dicycles of  $G$  form a unique outcome  $c$  of the game. Thus, the set of outcomes is  $A = \{a_1, \dots, a_p; c\}$ .

**Remark 1** In [10] a different approach was suggested (for  $n = 2$ ): each dicycle was treated as a separate outcome. Anyway, our main example contains only one dicycle.

To each player  $i \in I$  and outcome  $a \in A$  we assign a payoff (called in the literature also a reward, utility, or profit)  $u(i, a)$  of the player  $i \in I$  in case the outcome  $a \in A$  is realized. The corresponding mapping  $u : I \times A \rightarrow \mathbb{R}$  is called the payoff (reward, utility, or profit) function.

Since our main result is negative and related to chess-like games, we could restrict ourselves and the players to their strict preferences, instead of the real-valued payoffs. A complete order  $o_i$  over  $A$  is called the *preference* of the player  $i \in I$ ; and let  $o = (o_1, \dots, o_n)$  denote a *preference profile*.

A *backgammon-like game in the positional form* is the quadruple  $(G, D, o, v_0)$ , where  $G = (V, E)$  is a digraph,  $D : V = V_1 \cup \dots \cup V_n \cup V_T \cup V_R$  is a partition of the positions,  $o = (o_1, \dots, o_n)$  is a *preference profile*, and  $v_0$  is a fixed initial position. The triplet  $(G, D, v_0)$  is called a *positional game form*.

To define the *normal form* (of a chess-like game) let us introduce the strategies. A (*pure and stationary*) *strategy* of a player  $i \in I$  is a mapping that assigns a move  $(v, v')$  to each position  $v \in V_i$ . (In this paper we restrict ourselves and the players to their pure stationary strategies, so mixed and history dependent strategies will not even be introduced.) A set of  $n$  strategies  $s = \{s^i, i \in I\}$  is called a *strategy profile* or a *situation*. Each situation uniquely defines a *play*  $P(s)$  that begins in  $v_0$  and either ends in a terminal  $a \in V_T$  or cycles. In the latter case  $P(s)$  looks like a “lasso”: it consists of an initial part and a dicycle repeated infinitely. This is so, because a (pure stationary) strategy assigns the same move whenever a position is repeated and, hence, each situation  $s$  uniquely defines a move  $(v, v')$  in each non-terminal position  $v \in V \setminus V_T$ . Thus, we obtain a *game form*, that is, a mapping  $g : S \rightarrow A$ , where  $S = S_1 \times \dots \times S_n$  is the direct product of the sets  $S_i = \{s_1^i, \dots, s_{k_i}^i\}$  of strategies of all players  $i \in I$ . The *normal form* of a chess-like game  $(G, D, o, v_0)$  is defined as the pair  $(g, o)$ .

For the backgammon-like games each strategy profile  $s$  uniquely determines a Markovian chain, which assigns to each outcome  $a \in A$  (a terminal or an infinite play) a well defined limit probability  $p(s, a)$ . The payoff  $u(i, s)$  of a player  $i \in I$  in this situation  $s$  is defined as the expectation of the corresponding payoffs  $u(i, s) = \sum_{a \in A} p(s, a)u(i, a)$ .

A situation  $s \in S$  is called a *Nash equilibrium* (NE) if for each player  $i \in I$  and for each situation  $s'$  that may differ from  $s$  only in the coordinate  $i$ , the inequalities  $o_i(g(s)) \geq o_i(g(s'))$  and  $u(i, s) \geq u(i, s')$  hold in case of the chess- and backgammon-like games, respectively; or in other words, if no player  $i \in I$  can profit replacing his/her strategy  $s^i$  in  $s$  by a new strategy  $s'^i$ , provided the  $n - 1$  remaining players keep their strategies unchanged. Let us remark that the equality  $o_i(g(s)) = o_i(g(s'))$  may hold only when  $g(s) = g(s')$ , since the preference  $o_i$  is strict.

## 2 The main example

The positional and normal forms of the game announced in the title of the paper are presented below by Figure 1 and Table 1, respectively.

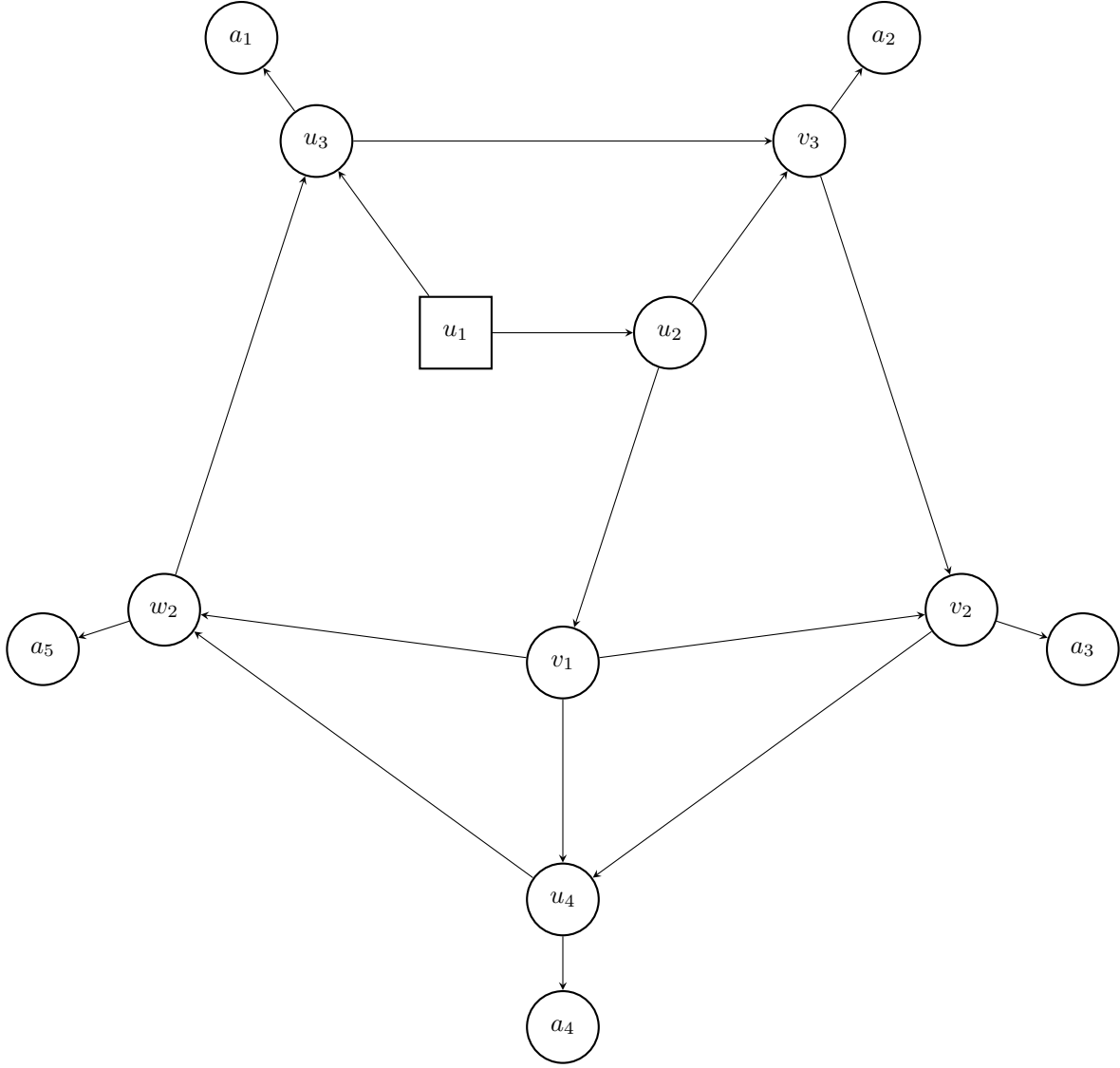


Figure 1: This figure represents our main example in the positional form.

Four players  $I = \{1, 2, 3, 4\}$  make decisions in eight non-terminal positions  $u_1, v_1; u_2, v_2, w_2; u_3, v_3$ , and  $u_4$ , respectively. The subscript is the number of the player who controls the corresponding position.

The initial position is  $u_1$ . There are five terminal positions  $a_j : j \in J = \{1, 2, 3, 4, 5\}$ .

There is a unique dicycle  $c$  and, thus, the set of outcomes is  $A = \{a_1, a_2, a_3, a_4, a_5; c\}$ .

The game has no NE whenever the preferences  $o_i$  of the players  $i \in I$  over the set of outcomes  $A$  agree with the following partial orders:

$O_1 : a_2 > a_4 > a_3 > a_1 > a_5$ ;  $O_2 : \min(a_1, c) > a_3 > \max(a_4, a_5) > \min(a_4, a_5) > a_2$ ;

$O_3 : \min(a_5, c) > a_1 > a_2 > \max(a_3, a_4)$ ;  $O_4 : \min(a_1, a_2, a_3, a_5) > a_4 > c$ .

		$s_1^4$								$s_2^4$							
		$s_1^2$	$s_2^2$	$s_3^2$	$s_4^2$	$s_5^2$	$s_6^2$	$s_7^2$	$s_8^2$	$s_1^2$	$s_2^2$	$s_3^2$	$s_4^2$	$s_5^2$	$s_6^2$	$s_7^2$	$s_8^2$
$s_1^3$	$s_1^1$	$c^4$	$c^4$	$a_3^2$	$a_3^{23}$	$a_5^2$	$a_5^2$	$a_3^2$	$a_3^{23}$	$a_4^2$	$a_4^{23}$	$a_3^1$	$a_3^3$	$a_4^{24}$	$a_4^{234}$	$a_3^1$	$a_3^3$
	$s_2^1$	$c^4$	$c^4$	$a_3^{23}$	$a_3^{23}$	$a_5^2$	$a_5^2$	$a_3^{23}$	$a_3^{23}$	$a_4^{23}$	$a_4^{23}$	$a_3^{13}$	$a_3^3$	$a_4^{234}$	$a_4^{234}$	$a_3^{13}$	$a_3^3$
	$s_3^1$	$c^4$	$c^4$	$a_3^{23}$	$a_3^{23}$	$a_5^2$	$a_5^2$	$a_5^{12}$	$a_3^{23}$	$a_4^2$	$a_4^{23}$	$a_4^{24}$	$a_3^3$	$a_4^{24}$	$a_4^{234}$	$a_4^{24}$	$a_3^3$
	$s_4^1$	$c^4$	$c^4$	$a_3^{23}$	$a_3^{23}$	$a_5^2$	$a_5^2$	$a_3^{23}$	$a_3^{23}$	$a_4^{23}$	$a_4^{23}$	$a_3^{13}$	$a_3^3$	$a_4^{234}$	$a_4^{234}$	$a_3^{13}$	$a_3^3$
	$s_5^1$	$c^4$	$c^4$	$a_3^{23}$	$a_3^{23}$	$a_5^2$	$a_5^2$	$a_5^{12}$	$a_3^{23}$	$a_4^{23}$	$a_4^{23}$	$a_3^{13}$	$a_3^3$	$a_5^{12}$	$a_4^{234}$	$a_5^{12}$	$a_3^3$
	$s_6^1$	$c^4$	$c^4$	$a_3^{23}$	$a_3^{23}$	$a_5^2$	$a_5^2$	$a_3^{23}$	$a_3^{23}$	$a_4^{23}$	$a_4^{23}$	$a_3^{13}$	$a_3^3$	$a_4^{234}$	$a_4^{234}$	$a_3^{13}$	$a_3^3$
$s_2^3$	$s_1^1$	$a_1^3$	$a_1^3$	$a_3^{23}$	$a_3^{23}$	$a_5^{12}$	$a_5^{12}$	$a_3^2$	$a_3^{23}$	$a_4^{24}$	$a_4^{234}$	$a_3^1$	$a_3^3$	$a_4^{24}$	$a_4^{234}$	$a_3^1$	$a_3^3$
	$s_2^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_1^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$
	$s_3^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_3^{23}$	$a_5^{12}$	$a_5^{12}$	$a_5^{12}$	$a_3^{23}$	$a_4^{24}$	$a_4^{234}$	$a_4^{24}$	$a_3^3$	$a_4^{24}$	$a_4^{234}$	$a_4^{24}$	$a_3^3$
	$s_4^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_1^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$
	$s_5^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_3^{32}$	$a_5^{12}$	$a_5^{12}$	$a_5^{12}$	$a_3^{23}$	$a_1^1$	$a_4^{234}$	$a_1^1$	$a_3^{23}$	$a_5^{12}$	$a_4^{234}$	$a_5^{12}$	$a_3^{23}$
	$s_6^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_1^1$	$a_1^3$	$a_1^3$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$
$s_3^3$	$s_1^1$	$a_2^{23}$	$a_2^{23}$	$a_3^1$	$a_2^2$	$a_5^{12}$	$a_2^{23}$	$a_3^1$	$a_2^2$	$a_4^{124}$	$a_4^2$	$a_3^1$	$a_2^2$	$a_4^{124}$	$a_2^2$	$a_3^1$	$a_2^2$
	$s_2^1$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$
	$s_3^1$	$a_2^{23}$	$a_2^{23}$	$a_2^{23}$	$a_2^2$	$a_5^1$	$a_2^{23}$	$a_5^1$	$a_2^2$	$a_4^{14}$	$a_4^2$	$a_4^{14}$	$a_2^2$	$a_4^{14}$	$a_2^2$	$a_4^{14}$	$a_2^2$
	$s_4^1$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$
	$s_5^1$	$a_2^{23}$	$a_2^{23}$	$a_2^{23}$	$a_2^2$	$a_5^1$	$a_2^{23}$	$a_5^1$	$a_2^2$	$a_2^{23}$	$a_2^2$	$a_2^{23}$	$a_2^2$	$a_5^1$	$a_2^2$	$a_5^1$	$a_2^2$
	$s_6^1$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$	$a_2^3$
$s_4^3$	$s_1^1$	$a_1^3$	$a_2^{23}$	$a_3^2$	$a_2^2$	$a_5^{12}$	$a_2^{23}$	$a_3^2$	$a_2^2$	$a_4^{24}$	$a_4^2$	$a_3^1$	$a_2^2$	$a_4^{24}$	$a_2^2$	$a_3^1$	$a_2^2$
	$s_2^1$	$a_1^3$	$a_1^{13}$	$a_1^1$	$a_1^1$	$a_1^3$	$a_1^{13}$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$
	$s_3^1$	$a_1^3$	$a_2^{23}$	$a_1^1$	$a_2^2$	$a_5^{12}$	$a_2^{23}$	$a_5^{12}$	$a_2^2$	$a_4^4$	$a_4^2$	$a_4^4$	$a_2^2$	$a_4^4$	$a_2^2$	$a_4^4$	$a_2^2$
	$s_4^1$	$a_1^3$	$a_1^{13}$	$a_1^1$	$a_1^1$	$a_1^3$	$a_1^{13}$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$
	$s_5^1$	$a_1^3$	$a_2^{23}$	$a_1^1$	$a_2^2$	$a_5^{12}$	$a_2^{23}$	$a_5^{12}$	$a_2^2$	$a_1^1$	$a_2^2$	$a_1^1$	$a_2^2$	$a_5^{12}$	$a_2^2$	$a_5^{12}$	$a_2^2$
	$s_6^1$	$a_1^3$	$a_1^{13}$	$a_1^1$	$a_1^1$	$a_1^3$	$a_1^{13}$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$	$a_1^1$

Table 1: This table represents our main example in the normal form.

The game form  $g : S \rightarrow A$ , in which  $S = S_1 \times S_2 \times S_3 \times S_4$  and  $A = \{a_1, a_2, a_3, a_4, a_5; c\}$ , is given by the four-dimensional table of size  $6 \times 8 \times 4 \times 2$ .

Player 1 has six strategies:

$s_1^1$ :  $(u_1, u_2), (v_1, v_2)$ ,  $s_2^1$ :  $(u_1, u_3), (v_1, v_2)$ ,  
 $s_3^1$ :  $(u_1, u_2), (v_1, u_4)$ ,  $s_4^1$ :  $(u_1, u_3), (v_1, u_4)$ ,  
 $s_5^1$ :  $(u_1, u_2), (v_1, w_2)$ ,  $s_6^1$ :  $(u_1, u_3), (v_1, w_2)$ ;

player 2 has eight strategies:

$s_1^2$ :  $(u_2, v_1), (v_2, u_4), (w_2, u_3)$ ,  $s_2^2$ :  $(u_2, v_3), (v_2, u_4), (w_2, u_3)$ ,  
 $s_3^2$ :  $(u_2, v_1), (v_2, a_3), (w_2, u_3)$ ,  $s_4^2$ :  $(u_2, v_3), (v_2, a_3), (w_2, u_3)$ ,  
 $s_5^2$ :  $(u_2, v_1), (v_2, u_4), (w_2, a_5)$ ,  $s_6^2$ :  $(u_2, v_3), (v_2, u_4), (w_2, a_5)$ ,  
 $s_7^2$ :  $(u_2, v_1), (v_2, a_3), (w_2, a_5)$ ,  $s_8^2$ :  $(u_2, v_3), (v_2, a_3), (w_2, a_5)$ ;

player 3 has four strategies:

$s_1^3$ :  $(u_3, v_3), (v_3, v_2)$ ,  $s_2^3$ :  $(u_3, a_1), (v_3, v_2)$ ,  $s_3^3$ :  $(u_3, v_3), (v_3, a_2)$ ,  $s_4^3$ :  $(u_3, a_1), (v_3, a_2)$ ;

finally, player 4 has two strategies:  $s_1^4(u_4, w_2)$ ,  $s_2^4(u_4, a_4)$ .

It is not difficult (although time consuming) to verify that the corresponding game has no NE whenever the preference profile  $o = \{o_1, o_2, o_3, o_4\}$  of the players  $I = \{1, 2, 3, 4\}$  agrees with the following partial orders:

$O_1 : a_2 > a_4 > a_3 > a_1 > a_5$ ;  $O_2 : \min(a_1, c) > a_3 > \max(a_4, a_5) > \min(a_4, a_5) > a_2$ ;

$O_3 : \min(a_5, c) > a_1 > a_2 > \max(a_3, a_4)$ ;  $O_4 : \min(a_1, a_2, a_3, a_5) > a_4 > c$ .

For every situation  $s = (s_{\ell_1}^1, s_{\ell_2}^2, s_{\ell_3}^3, s_{\ell_4}^4) \in S_1 \times S_2 \times S_3 \times S_4 = S$  the outcome  $g(s)$ , which is either a terminal  $a_j$  or the dicycle  $c$ , is shown in the entry  $(\ell_1, \ell_2, \ell_3, \ell_4)$  of the table. The upper indices indicate the players who can improve the situation  $s$ . Thus, a situation  $s$  is a NE if and only if the corresponding outcome has no upper indices. Since the table contains no such situation, the considered game has no NE.

### 3 Open ends

In the above example there are four players,  $n = 4$ , five terminals,  $p = 5$ , and there is only one dicycle.

In [8] it was shown that there is no chess-like NE-free game with  $p \leq 2$  terminals. In [11], Boros and Rand extended this result  $p \leq 3$ . Thus, only the case  $p = 4$  remains open.

It is known that every two-person chess-like game has a NE; see [8, 10], the last section of each paper. Thus, only the case  $n = 3$  remains open.

The proof for  $n = 2$  is simple and we repeat it here for convenience of the readers. It is based on the following important property of the two-person game forms, which seems not to be extendable for  $n > 2$ .

A two-person game form  $g$  is called (i) *Nash-solvable*, (ii) *zero-sum-solvable*, and (iii)  $\pm 1$ -*solvable* if the corresponding game  $(g, u)$  has at least one NE (i) for every payoff  $u = (u_1, u_2)$ ; (ii) for every payoff  $u = (u_1, u_2)$  such that  $u_1(a) + u_2(a) = 0$  for each outcome  $a \in A$ ; (iii) for every payoff  $u = (u_1, u_2)$  such that  $u_1(a) + u_2(a) = 0$  for each outcome  $a \in A$  and both  $u_1$  and  $u_2$  take only values  $+1$  or  $-1$ .

In fact, all three above properties of a game forms are equivalent. For (ii) and (iii) this was shown in 1970 by Edmonds and Fulkerson [14] and independently in [23]. Then, the list was extended by statement (i) in [24]; see also [25], where it also shown that a similar statement fails for the three-person game forms.

Thus, it is sufficient to prove  $\pm 1$ -solvability, rather than Nash-solvability, of the two-person chess-like games. Hence, we can assume that each outcome  $a \in A = V_T \cup \{c\}$  is either winning for player 1 and losing for player 2, or vice versa. Without any loss of generality, assume that  $c$  is winning for 1.

Then, let  $V_T = V_T^1 \cup V_T^2$  be the partition of all terminals into outcomes winning for players 1 and 2, respectively. Furthermore, let  $V^2 \subseteq V$  denote the set of all positions from which player 2 can enforce  $V_T^2$ ; in particular,  $V_T^2 \subseteq V^2$ . Finally, let us set  $V^1 = V \setminus V^2$ ; in particular,  $V_T^1 \subseteq V^1$ .

By the above definitions, in every position  $v \in V_1 \cap V^1$  player 1 can stay out of  $V^2$ , that is, (s)he has a move  $(v, v')$  such that  $v' \in V^1$ . Let us fix a strategy  $s_0^1$  that chooses such a move in each position  $v \in V_1 \cap V^1$  and any move in  $v \in V_1 \cap V^2$ . Then, for any  $s^2 \in S_2$ , the outcome  $g(s_0^1, s^2)$  is winning for player 1 whenever the initial position  $v_0$  is in  $V^1$ . Indeed, either  $g(s_0^1, s^2) \in V_T^1$ , or  $g(s_0^1, s^2) = c$ ; in both cases player 1 wins. Thus, player 1 wins when  $v_0 \in V^1$  and player 2 wins when  $v_0 \in V^2$ ; in each case a saddle point exists.

In [30], we conjectured that there is a NE-free chess-like game satisfying the following extra condition

- (C) outcome  $c$  is worse than any terminal outcome  $a \in V_T$  for each player  $i \in I$ .

This conjecture remains open. Such an example, if exists, would strengthen simultaneously the example of the present paper and the main example of [30]; see Figure 2 and Table 2 there.

**Remark 2** In [3], 16 problems related to Nash-solvability, subgame perfect and not, of the chess-like and backgammon-like games were considered, under the assumption (C) and without it. For 15 of these problems, condition (C) appears “irrelevant”, that is, either Nash-solvability holds, even without (C), or even with (C), it fails. (Yet, in the latter case, the size of the corresponding example may increase significantly.) Based on these observations, we conjectured in [30] that the same will happen for the last 16th case, which is the subject of the present paper. An example might be similar to one in Figure 1, but with a much larger digraph. Several interpretations of assumption (C) were suggested Remark 2 and Section 2.1.1 of [8].

Finally, it follows from the main result of [8] that there is no NE-free  $n$ -person chess-like game in which each player controls a unique position. In the above example, the players 1, 2, 3, and 4 control 2, 3, 2, and 1 positions, respectively. It remains open, if there is a chess-like NE-free game in which each player controls, say, at most two positions.

### 4 Related results on Nash-solvability

In the next two subsections we recall two large families of games with perfect information that are known to be Nash-solvable in pure stationary uniformly optimal strategies.

#### 4.1 Acyclic $n$ -person backgammon-like games with perfect information

In 1950 Nash introduced his concept of equilibrium for the normal form  $n$ -person games [40, 41]. Soon after, Kuhn [34, 35] and Gale [20] suggested the backward induction procedure and proved that any finite acyclic chess-like game with perfect information has a NE in pure stationary strategies; moreover,

the obtained NE is *subgame perfect*, that is, the same strategy profile is a NE with respect to any initial position. The authors restricted themselves to the chess-like games on finite arborescence (directed trees) but, in fact, backward induction can be easily extended to the backgammon-like games on the finite digraphs without dicycles (so called *acyclic digraphs* or *DAGs*). Yet, acyclicity is a crucial assumption and cannot be waved.

For any integer  $k \geq 2$  let us introduce a digraph  $G_k$  that consists of  $k$  terminals  $a_1, \dots, a_k$ , the directed  $k$ -cycle  $C_k$  on the  $k$  non-terminal vertices  $v_1, \dots, v_k$ , and the perfect matching  $(v_j, a_j), j = 1, \dots, k$  between these vertices and the terminals.

The existence of a subgame perfect NE fails already for  $k = 2$  [1]. Let players 1 and 2 control vertices  $v_1$  and  $v_2$  and have the preferences,  $o_1 : (c > a_1 > a_2)$  and  $(a_1 > a_2 > c)$ , respectively. It is easy to verify that a NE exists for any given initial position,  $v_1$  or  $v_2$ , but no strategy profile is a NE with respect to both simultaneously. Let us notice that the preferences are not opposite (both players prefer  $a_1$  to  $a_2$ ), while  $c$  is the worst outcome for player 2 and the best one for 1.

A similar example exists even if in addition we require (C): the dicycle is worse than each terminal for both players. Consider digraph  $G_6$  in which players 1 and 2 control the odd and even positions  $(a_1, a_3, a_5$  and  $a_2, a_4, a_6)$  respectively. It was shown in [3] that there exists no subgame perfect NE whenever

$$o_1 : a_6 > a_5 > a_2 > a_1 > a_3 > a_4 > c; \quad o_2 \in O_2 : \{a_3 > a_2 > a_6 > a_4 > a_5 > c, a_6 > a_1 > c\}.$$

Note that there is no such example for  $G_4$ . A similar example for a three-person game was constructed in [3]. The players 1, 2, 3 control, respectively, the positions  $v_1, v_2, v_3$  of  $G_3$  and have the preferences:

$$o_1 : a_2 > a_1 > a_3 > c; \quad o_2 : a_3 > a_2 > a_1 > c; \quad o_3 : a_1 > a_3 > a_2 > c.$$

In other words, for all players:  $c$  is the worst outcome, in accordance with (C); it is better if the previous player terminate; it is still better to terminate himself; finally, the best if the next player terminates.

In [7], these results were strengthened as follows. It was demonstrated that a subgame perfect NE may fail to exist not only in the pure but even in the mixed strategies. The corresponding examples are based on the same positional game forms,  $G_6$  for  $n = 2$  and  $G_3$  for  $n = 3$ , but the above preference profiles are replaced by some carefully chosen payoffs, which agree with the corresponding preferences.

The above examples imply that, for any  $n \geq 2$ , an  $n$ -person *backgammon-like* game, even with a fixed initial position, may have no NE. Given a chess-like game  $(G = (V, E), D, o)$ , in which no initial position is fixed yet, add to it a new position  $v_0$  and the move  $(v_0, v)$  from  $v_0$  to each non-terminal position  $v \in V \setminus V_T$ . Furthermore, assign to  $(v_0, v)$  a non-negative probability  $p_v \geq 0$  such that  $\sum_{v \in V} p_v = 1$ . Denote by  $(G', D, o, v_0)$  the obtained backgammon-like game form, in which  $v_0$  is the initial position; see Figure 1 in [3]. It was shown in [3] that

- (i) if  $s$  is a subgame perfect NE in  $(G, D, o)$  then  $s$  is a NE in  $(G', D, o, v_0)$  for any  $p_v$ ;
- (ii) if  $s$  is a NE in  $(G', D, o, v_0)$  and  $p_v > 0 \forall v \in V \setminus V_T$ , then  $s$  is a subgame perfect NE in  $(G_k, D, o)$ .

These results imply that Nash-solvability of  $(G', D, v_0)$  is equivalent with subgame perfect Nash-solvability of  $(G, D)$ . As we know, the latter property may fail for  $G = G_k$  for any  $n \geq 2$ . Thus, for any  $n \geq 2$ , an  $n$ -person backgammon-like game  $(G'_k, D, v_0, o)$ , in which  $v_0$  is a unique position of chance, may have no NE.

## 4.2 Two-person zero-sum games with perfect information

According to the previous subsection, in the presence of dicycles, backward induction fails, in general. Yet, it can be modified (and thus saved) in case of the two-person zero-sum games (that is, when  $I = \{1, 2\}$  and  $u(1, a) + u(2, a) = 0$  for any outcome  $a \in A$ , or two preferences  $o_1$  and  $o_2$  of the players 1 and 2 over  $A$  are opposite).

For example, the recent paper [22] shows how to solve by backward induction a two-person zero-sum game that is “acyclic”, but the players can pass; in other words, the corresponding digraph contains a loop at each vertex, but no other dicycles.

A general linear time algorithm solving any two-person zero-sum chess-like game, by a modified backward induction, was suggested in [2] and independently in [3]. In contrast, no polynomial algorithm is known for the two-person zero-sum backgammon-like games [12, 13]. However, it is well-known that subgame perfect saddle points in stationary strategies exist in this case and even in much more general cases considered below.

In fact, studying two-person zero-sum chess-like games began long before the backward induction was suggested in early fifties by [34, 35, 20]. Zermelo gave his seminal talk on solvability of chess in pure strategies [46] as early as in 1912. Later, König [33] and Kalman [32] strengthen this result showing that there exist pure stationary uniformly optimal strategies producing a subgame perfect saddle point, in any two-person zero-sum chess-like game. The reader can find more detailed surveys in [45, 17, 42, 2]; see also [9, 18, 19, 28].

The chess-like and backgammon-like games, considered in this note, by the definition, are *transition-free*. There is a much more general class: *stochastic games with perfect information* in which a *transition payoff*  $r_i(u, v)$  is defined for every move  $(u, v)$  and for each player  $i \in I$ . Gillette, in his seminal paper [21], introduced the *mean (or average)* effective payoff for these games and proved the existence of subgame perfect saddle point in the uniformly optimal stationary strategies for the two-person zero-sum case. The proof is pretty complicated. It is based on the Tauberian theory and, in particular, on the Hardy-Littlewood theorem [31]. (In [21], the conditions of this theorem were not accurately verified and the flaw was corrected in twelve years by Liggett and Lippman in [37].)

Stochastic games with perfect information can be viewed as *backgammon-like* games with transition payoffs. More precisely, these two classes are polynomially equivalent [4]. Interestingly, the corresponding two-person zero-sum *chess-like* games (with transition payoffs but without random moves) so-called *cyclic mean-payoff games*, appeared only 20-30 years later, introduced for the complete bipartite digraphs by Moulin [38, 39], for any bipartite digraphs by Ehrenfeucht and Mycielski [15, 16], and for arbitrary digraphs by Gurvich, Karzanov, and Khachiyan [29]. Again, the existence of a saddle point in the pure stationary uniformly optimal strategies was proven for the two-person zero-sum case.

This result cannot be extended to the non-zero-sum case. In [25], a cyclic mean payoff two-person NE-free game was constructed on the complete bipartite  $3 \times 3$  digraph with symmetric payoffs. (The corresponding normal form game is a  $27 \times 27$  bimatrix.) It was shown in [27] that this example is, in a sense, minimal, namely, a NE always exists for the games on the complete  $(2 \times \ell)$  bipartite digraphs.

A general family of the so-called *k-total effective payoffs* was recently introduced in [6] for any non-negative integer  $k$  such that the 0-total one is the mean effective payoff, while the 1-total one is the total effective payoff introduced earlier by Thuijsman and Vrieze in [43, 44]. The existence of a saddle point in uniformly optimal pure stationary strategies for the two-person zero-sum chess-like games with the  $k$ -total effective payoff was proven for all  $k$  in [6]. If this result can be extended to the backgammon-like games is an open problem. Yet, for  $k \leq 1$  the answer is positive. As was already mentioned, for  $k = 0$  it was proven long ago. For  $k = 1$  the result was first obtained in [43, 44], see also [6].

However, it cannot be extended to the non-zero-sum case. In particular, a NE (in pure stationary strategies) may fail to exist already in a two-person chess-like game. For  $k = 0$  the example was given in [25]. Furthermore, in [6], a simple embedding of the  $(k - 1)$ -total payoff games into the  $k$ -total ones was constructed. Thus, the example of [25] works for all  $k$ .

**Acknowledgements:** The authors are thankful to Endre Boros, Konrad Borys, Khaled Elbassioni, and Gabor Rudolf for helpful discussions.

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